Generalized Inner-Outer Factorizations in non commutative Hardy Algebras

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Abstract

Let $H^\infty(E)$ be a non commutative Hardy algebra, associated with a W^*-correspondence E. In this paper we construct factorizations of inner-outer type of the elements of $H^\infty(E)$ represented via the induced representation, and of the elements of its commutant. These factorizations generalize the classical inner-outer factorization of elements of $H^\infty(\mathbb{D})$. Our results also generalize some results that were obtained by several authors in some special cases.

1 Introduction

In this work we describe the general version of the inner-outer factorization in non commutative Hardy algebras. Recall that the Hardy algebra $H^\infty(\mathbb{D})$ is identified with the algebra $H^\infty := H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, where $H^2 = H^2(\mathbb{T})$ is the Hardy Hilbert space, and we consider H^∞ as the algebra of multiplication operators M_ϕ acting on the Hilbert space H^2 by $f \mapsto \phi f$. Then the function $\Theta \in H^\infty$ is called inner if the operator M_Θ is isometric and the function $g \in H^\infty$ is called outer if the operator M_g has a dense range in H^2. The classical theorem says that every $f \in H^\infty$ admits a unique inner-outer factorization $f = f_i f_o$, where f_i is an inner function, called also the inner part of f, and f_o is an outer, called the outer part of f. Analogous factorizations hold in the Hardy spaces H^p, $p \geq 1$. In particular, every $f \in H^2$ admits an inner-outer factorization o$f = \Theta g$, with $f_i \in H^\infty$ and $f_o \in H^2$. Further, any z-invariant subspace of the form $\mathcal{M}_f = \vee\{z^n f : n = 0, 1, ...\}$ has the representation $\mathcal{M}_f = f_i H^2$. The classical Beurling' theorem says that every z-invariant subspace $\mathcal{M}$ has a representation $\mathcal{M} = \Theta H^2$ for a suitable inner function Θ, [2]. A full treatment of the classical theory both from the function theoretic and the operator theoretic point of view can be found in [12] and [13].

Before we introduce the non commutative Hardy algebras note that the classical algebra $H^\infty(\mathbb{D})$ can be viewed as the ultraweak closure of the operator algebra generated by the unilateral shift on the Hilbert space $l^2 = l^2(\mathbb{Z}_+)$. In [16] this was generalized by G. Popescu to the ultraweakly closed non commutative operator algebras generated by d shifts, $d \geq 1$, denoted $\mathcal{F}^\infty$. In [1], [17], [18], [19] Arias and Popescu developed the theory of inner-outer factorization in $\mathcal{F}^\infty$. In [3] Davidson and Pitts developed analogous theory, with some differences, in the context of the free semigroup algebra $\mathcal{L}_d$, which, in fact, coincides with $\mathcal{F}^\infty$. In [4] Kribs and Power considered the case of free semigroupoids algebras $\mathcal{L}_G$, and developed the theory of inner-outer factorization in these algebras.

In this work we develop our version of the inner-outer factorization in non commutative Hardy algebras $H^\infty(E)$ associated with a given W^*-correspondence E. These algebras

were introduced in 2004 by P. Muhly and B. Solel in [10] (see also [9]), and generalize the classical Hardy algebra H^∞, the algebra $\mathcal{F}^\infty$ of Popescu, free semigroups algebras, free semigroupoids algebras and some others.

Let E be a W^*-correspondence over a W^*-algebra M, ([6], [14]), that is a right Hilbert W^*-module E over M, which is made into a M-M - bimodule by some $*$-homomorphism $\phi : M \to \mathcal{L}(M)$, where $\mathcal{L}(M)$ is the algebra of all the adjointable operators on E. This W^*-correspondence defines another W^*-correspondence $\mathcal{F}(E)$ over the same algebra M, which is defined to be the direct sum $M \oplus E \oplus E^{\otimes 2} \oplus ...$ of the internal tensor powers of E. $\mathcal{F}(E)$ is called the full Fock space and in fact is a W^*-correspondence with the left action of M denoted by ϕ_∞, which is a natural extension of ϕ to a representation of M in the algebra of adjointable operators on $\mathcal{F}(E)$. Note that the space $\mathcal{L}(E)$, for any W^*-correspondence E, is a W^*-algebra. The non commutative Hardy algebra of a correspondence E is by definition the weak*-closure in $\mathcal{L}(\mathcal{F}(E))$ of the algebra spanned by the operators of the form T_ξ, $\xi \in E$, where $T_\xi(\eta) := \xi \otimes \eta$ and $\phi_\infty(a)$, $a \in M$. In fact Muhly and Solel defined this Hardy algebra as the weak* closure of the noncommutative tensor algebra $\mathcal{T}_+(E)$. The algebra $\mathcal{T}_+(E)$ was defined first in [9] as the norm closed (nonselfadjoint) algebra spanned by the same set of generators, and it generalizes the noncommutative disc algebra $\mathcal{A}_n$ of Popescu, which in its turn is a noncommutative generalization of the classical disc algebra.

In this work we view the algebra $H^\infty(E)$ as acting on a Hilbert space via an induced representation ρ and write it $\rho(H^\infty(E))$. Thus we consider the questions of inner-outer factorization for the case of the Hardy algebra $\rho(H^\infty(E))$. A key tool that we will need and use here is the general version of Wold decomposition proved first in [8]. We start with the inner-outer factorization of a vector of the underlying Hilbert space, and then we obtain the inner-outer factorization of an element of the commutant of the algebra $\rho(H^\infty(E))$. Here we use that fact that, as in the abstract theory of shifts, an inner operator is a partial isometry in the commutant of the algebra, generated by a shift. Further, we translate the Beurling theorem of Muhly and Solel in [8] to our language. It follows from the concept of duality for W^*-correspondences, developed in [10], every algebra $\rho(H^\infty(E))$ can be thought of as the commutant of the Hardy algebra of another correpondence, called the dual of E. Using this concept we construct factorization of an element of $\rho(H^\infty(E))$ which holds in our setup.

2 Preliminaries and Setting

We start by recalling the notion of a W^*-correspondence. For a general theory of Hilbert C^*- and W^*-modules we use [5], [6] and the original paper [14]. Here we only note that by a Hilbert W^*-module we always mean a self dual module over a W^*-algebra (see [6, Ch. 3]).

Let $\phi : M \to \mathcal{L}(E)$ be a normal $*$-homomorphism. In what follows we always assume that ϕ is unital. Then we obtain on E the structure of a bimodule over M. We shall call it a

W^*-correspondence over the W^*-algebra M. More generally, let N and M be W^*-algebras and let E be a Hilbert M-module. Assume that we are given a left action of N on E, that is, we are given normal $*$-homomorphism $\phi : N \to \mathcal{L}(E)$. This homomorphism can be regarded as a "generalized homomorphism" from N to M. Such an N-M-bimodule E will be called a correspondence from N to M. Every W^*-correspondence E has the structure of a dual Banach space [14]. This topology is usually called the σ-topology, [10].

Every Hilbert space H, where the inner product is taken to be linear in the second variable, is a W^*-module and a W^*-correspondence over $\mathbb{C}$ in a natural way.

Let E and F be W^*-correspondences over W^*-algebras M and N respectively. The left action of M on E will be denoted as usual by ϕ and the left action of N on F by ψ, thus, $\psi : N \to \mathcal{L}(F)$ is a normal $*$-homomorphism.

Definition 2.1. An isomorphism of E and F is a pair (σ, Φ) where

1) $\sigma : M \to N$ is an isomorphism of W^*-algebras;

2) $\Phi : E \to F$ is a vector space isomorphism preserving the σ-topology, and which is also

(a) a bimodule map, $\Phi(\phi(a)xb) = \psi(\sigma(a))\Phi(x)\sigma(b)$, $x \in E$ $a, b \in M$, and

(b) Φ "preserves" the inner product, $\langle \Phi(x), \Phi(y)\rangle = \sigma(\langle x, y\rangle)$, $x, y \in E$.

Let E be a W^*-correspondence over a W^*-algebra M with a left action defined as usual by a normal $*$-homomorphism ϕ. For each $n \geq 0$, let $E^{\otimes n}$ be the self-dual internal tensor power (balanced over ϕ, [10]). So, $E^{\otimes n}$ itself turns out to be a W^*-correspondence in a natural way, with the left action $\xi \mapsto \phi_n(a)\xi = (\phi(a)\xi_1) \otimes ... \otimes \xi_n$, $\xi = \xi_1 \otimes ...\xi_n \in E^{\otimes n}$, and with an M-valued inner product as in the internal tensor product construction. For example, on $E^{\otimes 2} = E \otimes_\phi E$, we define

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = \langle \xi_2, \phi(\langle \xi_1, \eta_1\rangle)\eta_2\rangle.$$

We form the full Fock space $\mathcal{F}(E) = \sum_{n\geq 0}^{\oplus} E^{\otimes n}$, where $E^{\otimes 0} = M$ and the direct sum taken in the ultraweak sense (see [14]). This is a W^*-correspondence with left action given by $\phi_\infty : M \to \mathcal{L}(\mathcal{F}(E))$, where $\phi_\infty(a) = \sum_{n\geq 0} \phi_n(a)$. The M-valued inner product on $\mathcal{F}(E)$ is defined in an obvious way.

For each $\xi \in E$ and each $\eta \in \mathcal{F}(E)$, let $T_\xi : \eta \mapsto \xi \otimes \eta$ be a creation operator on $\mathcal{F}(E)$. Clearly, $T_\xi \in \mathcal{L}(\mathcal{F}(E))$.

Definition 2.2. Given a W^*-correspondence E over a W^*-algebra M.

1) The norm closed subalgebra in $\mathcal{L}(\mathcal{F}(E))$, generated by all creation operators T_ξ, $\xi \in E$, and all operators $\phi_\infty(a)$, $a \in M$, is called the tensor algebra of E. It is denoted by $\mathcal{T}_+(E)$.

2) The Hardy algebra $H^\infty(E)$ is the ultra-weak closure of $\mathcal{T}_+(E)$.

When $M = E = \mathbb{C}$ then $\mathcal{F}(E) = l^2(\mathbb{Z}_+)$. The algebra $\mathcal{T}_+(\mathbb{C})$ is the algebra of analytic Toeplitz operators with continuous symbols, so it can be identified with the disc algebra

$A(\mathbb{D})$. The algebra $H^\infty(\mathbb{C})$, in this case, is $H^\infty(\mathbb{D})$. If $M = \mathbb{C}$ and we take $E = \mathbb{C}^n$, an n-dimensional Hilbert space, then $\mathcal{T}_+(\mathbb{C}^n)$ is the non commutative disc algebra $\mathcal{A}_n$, studied by Popescu and others, and $H^\infty(\mathbb{C}^n) = \mathcal{F}_n^\infty$, the Hardy algebra of Popescu. This algebra can be identified with the free semigroup algebra $\mathcal{L}_n$ studied by Davidson and Pitts.

Let $\pi : M \to B(H)$ be a normal representation of a W^*-algebra M on a Hilbert space H and let E be a W^*-correspondence over M. As it can be easy verified, the W^*-internal tensor product $E \otimes_\pi H$ is a Hilbert space. The representation $\pi^E : \mathcal{L}(E) \to B(E \otimes_\pi H)$ defined by

$$\pi^E : S \mapsto S \otimes I_H, \quad \forall S \in \mathcal{L}(E).$$

is called the induced representation (in the sense of Rieffel). If π is a faithful normal representation then π^E maps $\mathcal{L}(E)$ into $B(E \otimes_\pi H)$ homeomorphically with respect to the ultraweak topologies, [10, Lemma 2.1].

The image of $H^\infty(E)$ under an induced representation is defined as follows. Let $\pi : M \to B(H)$ be a faithful normal representation. For a W^*-correspondence E over M let $\pi^{\mathcal{F}(E)}$ be the induced representation of $\mathcal{L}(\mathcal{F}(E))$ in $B(\mathcal{F}(E) \otimes_\pi H)$. Then the induced representation of the Hardy algebra $H^\infty(E)$ is the restriction

$$\rho := \pi^{\mathcal{F}(E)}|_{H^\infty(E)} : H^\infty(E) \to B(\mathcal{F}(E) \otimes_\pi H). \tag{1}$$

This restriction is an ultraweakly continuous representation of $H^\infty(E)$ and the image $\rho(H^\infty(E))$ is an ultraweakly closed subalgebra of $B(\mathcal{F}(E) \otimes_\pi H)$. We shall refer to ρ as the representation induced by π. Later, when we discuss several representation of $H^\infty(E)$ that are induced by different representations π, σ etc. of M, we shall write ρ_π, ρ_σ etc.

So, $\rho(H^\infty(E))$ acts on $\mathcal{F}(E) \otimes_\pi H$ and ρ is defined by

$$\rho : X \mapsto X \otimes I_H, \ \forall X \in H^\infty(E).$$

Note that the notion of the induced representation generalizes the notion of pure isometry (i.e. an isometry without a unitary part) in the theory of a single operator.

We will frequently use the following result of Rieffel [20, Theorem 6.23]. The formulation here is in a form convenient for us ([8, p. 853]).

Theorem 2.3. . *Let E be a W^*-correspondence over the algebra M and $\pi : M \to B(H)$ be a normal faithful representation of M on the Hilbert space H. Then the operator R in $B(E \otimes_\pi H)$ commutes with $\pi^E(\mathcal{L}(E))$ if and only if R is of the form $I_E \otimes X$, where $X \in \pi(M)'$, i.e., $\pi^E(\mathcal{L}(E))' = I_E \otimes \pi(M)'$.*

2.1 Covariant representations.

Definition 2.4. Let E be a W^*-correspondence over a W^*-algebra M.

(1) By a covariant representation of E, or of the pair (E, M), on a Hilbert space H, we mean a pair (T, σ), where $\sigma : M \to B(H)$ is a nondegenerate normal $*$-homomorphism,

and T is a bimodule (with respect to σ) map $T : E \to B(H)$, that is a linear map such that $T(\xi a) = T(\xi)\sigma(a)$ and $T(\phi(a)\xi) = \sigma(a)T(\xi)$, $\xi \in E$ and $a \in M$. We require also that T will be continuous with respect to the σ-topology on E and the ultraweak topology on $B(H)$.

(2) The representation (T, σ) is called (completely) bounded, (completely) contractive, if so is the map T. For a completely contractive covariant representation we write also c.c.c.r.

The operator space structure on E to which this definition refers is the one which comes from the embedding of E into its so-called linking algebra $\mathfrak{L}(E)$, see [9].

In this work we will consider only isometric covariant representations. A covariant representation (V, σ) is said to be isometric if $V(\xi)^*V(\eta) = \sigma(\langle \xi, \eta \rangle)$, for every $\xi, \eta \in E$. Every isometric covariant representation (V, π) of E is completely contractive, see [9, Corollary 2.13].

As an important example let $\rho = \pi^{\mathcal{F}(E)}|_{H^\infty(E)}$ be an induced representation of the Hardy algebra $H^\infty(E)$. For the representation σ set

$$\sigma = \pi^{\mathcal{F}(E)} \circ \phi_\infty,$$

and set

$$V(\xi) = \pi^{\mathcal{F}(E)}(T_\xi), \quad \xi \in E.$$

Definition 2.5. The pair (V, σ) is called the covariant representation induced by π, or simply the induced covariant representation (associated with ρ).

It is easy to check that (V, σ) in the above definition is isometric, hence, is completely contractive.

Let (T, σ) be a c.c.c.r. of (E, M) on the Hilbert space H as above. With each such representation we associate the operator $\tilde{T} : E \otimes_\sigma H \to H$, that on the elementary tensors is defined by

$$\tilde{T}(\xi \otimes h) := T(\xi)(h).$$

$\tilde{T}$ is well defined since $T(\xi a) = T(\xi)\sigma(a)$. In [9] Muhly and Solel show that the properties of $\tilde{T}$ reflect the properties of the covariant representation (T, σ). They proved that (α) $\tilde{T}$ is bounded iff T is completely bounded, and in this case $\|T\|_{cb} = \|\tilde{T}\|$; (β) $\tilde{T}$ is contractive iff T is completely contractive; and (γ) $\tilde{T}$ is an isometry iff (T, σ) is an isometric representation. A simple calculation gives us the intertwining relation

$$\tilde{T}\sigma^E \circ \phi(a) = \tilde{T}(\phi(a) \otimes I_H) = \sigma(a)\tilde{T}, \quad \forall a \in A. \tag{2}$$

In the following theorem we collect two basic facts concerning the theory of representations of W^*-correspondences and of their tensor algebras.

Theorem 2.6. *([10, Lemma 2.5 and Theorem 2.9]) Let E be any W^*-correspondence over an algebra M. Then*

1) There is a bijective correspondence $(T,\sigma) \leftrightarrow \tilde{T}$ between all c.c.c.r. (T,σ) of E on a Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \to H$ that satisfy the relation (2). Let $\tilde{T} : E \otimes_\sigma H \to H$ be a contraction that satisfies the relation (2). Then the associated covariant representation is the pair (T,σ), where T is defined by $T(\xi)h := \tilde{T}(\xi \otimes h)$, $h \in H$ and $\xi \in E$.

2) Let E be a W^*-correspondence over the algebra M and let (T,σ) be a c.c.c.r. of (E,M) on a Hilbert space H. Then for every such representation there exists a completely contractive representation $\rho : \mathcal{T}_+(E) \to B(H)$ such that $\rho(T_\xi) = T(\xi)$ for every $\xi \in E$ and $\rho(\phi_\infty(a)) = \sigma(a)$ for every $a \in M$. Moreover, the correspondence $(T,\sigma) \leftrightarrow \rho$ is a bijection between the set of all c.c.c.r. of E and all completely contractive representations of $\mathcal{T}_+(E)$ whose restrictions to $\phi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

Restricting our attention to isometric covariant representation, we have the following.

Lemma 2.7. *([8, Lemma 2.1].) Let (V,σ) be any isometric covariant representation of the W^*-correspondence E on a Hilbert space H. Then the associated isometry $\tilde{V} : E \otimes_\sigma H \to H$ is an isometry that satisfy the relation $\tilde{V}\sigma^E \circ \phi(a) = \sigma(a)\tilde{V}$, $\forall a \in M$, and with range equal to the closed linear span of $\{V(\xi)h : \xi \in E, a \in M\}$. Conversely, given an isometry $\tilde{V} : E \otimes_\sigma H \to H$ that satisfies the above intertwining relation, then the associated covariant representation is the pair (V,σ), where V is defined by $V(\xi)h := \tilde{V}(\xi \otimes h)$, $h \in H$ and $\xi \in E$.*

The representation ρ of $\mathcal{T}_+(E)$ that corresponds to the covariant representation (T,σ) is called the integrated form of (T,σ) and denoted by $\sigma \times T$. In its turn, the representation (T,σ) is called the desintegrated form of ρ. Preceding results show that, given a normal representation σ of M, the set of all completely contractive representations of $\mathcal{T}_+(E)$ whose restrictions to $\phi_\infty(M)$ is given by σ can be parameterized by the contractions $\tilde{T} \in B(E \otimes_\sigma H, H)$, that satisfy the relation (2).

In this notations, the induced representation ρ_π is an integrated form of the (V,σ), the covariant induced representation of E from Definition 2.5.

In [10] it was shown that, if the representation (T,σ) of (E,M) is such that $\|\tilde{T}\| < 1$, then the integrated form $\sigma \times T$ extends from $\mathcal{T}_+(E)$ to an ultraweakly continuous representation of $H^\infty(E)$. For a general (T,σ), the question when such an extention is possible is more delicate, see about this [11].

Let (V,σ) be an isometric covariant representation of a general W^*-correspondence E on a Hilbert space G. For every $n \geq 1$ write $(V^{\otimes n},\sigma)$ for the isometric covariant representation of $E^{\otimes n}$ on the same space G defined by the formula $V^{\otimes n}(\xi_1 \otimes ... \otimes \xi_n) = V(\xi_1)\cdots V(\xi_n)$, $n \geq 1$. The associated isometric operator $\tilde{V}_n : E^{\otimes n} \otimes_\sigma G \to G$ (which is called the generalized power of $\tilde{V}$), satisfies the identity $\tilde{V}_n\sigma^{E^{\otimes n}} \circ \phi_n = \tilde{V}_n(\phi_n \otimes I_G) = \sigma\tilde{V}_n$. In this notation $\tilde{V} = \tilde{V}_1$.

For each $k \geq 0$ write G_k for $\bigvee\{V(\xi_1)\cdots V(\xi_k)g : \xi_i \in E, g \in G\}$ (with $G_0 = G$). Clearly, $G_k = \overline{\tilde{V}_k(E^{\otimes k} \otimes_\sigma G_0)}$. Write R_k the projection of G_0 onto G_k and let $P_k = R_k - R_{k+1}$ and $R_\infty = \wedge_k R_k$. Thus, $R_k = \sum_{l \geq k} P_l + R_\infty$ is a projection of G_0 onto G_k and $R_0 = I_{G_0}$. According to [8], the formula $L(x) = \tilde{V}_1(I_1 \otimes x)\tilde{V}_1^*$ defines a normal endomorphism of the commutant $\sigma(M)'$ and its n-th iterate is $L^n(x) = \tilde{V}_n(I_n \otimes x)\tilde{V}_n^*$. Here $I_n = I_{E^{\otimes n}}$. Simple calculation shows that $L^n(P_m) = P_{n+m}$ and $R_n = \tilde{V}_n\tilde{V}_n^* = L^n(I)$.

An isometric covariant representation (V, σ) is called fully coisometric if $R_1 = L(I_{G_0}) = I_{G_0}$. Muhly and Solel proved the following Wold decomposition theorem ([8, Theorem 2.9]):

Theorem 2.8. *Let (V, σ) be an isometric covariant representation of W^*-correspondence E on a Hilbert space G_0. Then (V, σ) decomposes into a direct sum $(V_1, \sigma_1) \oplus (V_2, \sigma_2)$ on $G_0 = H_1 \oplus H_2$, where $(V_1, \sigma_1) = (V, \sigma)|_{H_1}$ is an induced representation and $(V_2, \sigma_2) = (V, \sigma)|_{H_2}$ is fully coisometric. Further, this decomposition is unique in the sense that if $K \subseteq G_0$ reduces (V, σ) and the restriction $(V, \sigma)|_K$ is induced (resp. fully coisometric) then $K \subseteq H_1$ (resp. $K \subseteq H_2$).*

From this theorem it follows immediately that (V, σ) is an induced representation if and only if $R_\infty = \wedge_k R_k = 0$.

With (V, σ) we may associate the "shift" $\mathfrak{L}$, that acts on the lattice of $\sigma(M)$-invariant subspaces of G, and is defined as a geometric counterpart of the endomorphism L. In a more details, let $\mathcal{M} \in Lat(\sigma(M))$, then we set

$$\mathfrak{L}(\mathcal{M}) := \bigvee\{V(\xi)k : \xi \in E, k \in \mathcal{M}\}. \tag{3}$$

The s-power $\mathfrak{L}^s(\mathcal{M})$ is defined in the obvious way (with $\mathfrak{L}^0(\mathcal{M}) = \mathcal{M}$)).

The subspace $\mathcal{M} \in Lat(\sigma(M))$, as well as its projection $P_\mathcal{M} \in \sigma(M)'$, is called wandering with respect to (V, σ), if the subspaces $\mathfrak{L}^s(\mathcal{M})$, $s = 0, 1, ...,$ are mutually orthogonal. Write σ' for the restriction $\sigma|_\mathcal{M}$, where $\mathcal{M}$ is wandering. Then the Hilbert space $E^{\otimes s} \otimes_{\sigma'} \mathcal{M}$ is isometrically isomorphic (under the generalised power $\tilde{V}_s$) to $\mathfrak{L}^s(\mathcal{M})$. Hence, we obtain an isometric isomorphism

$$\mathcal{F}(E) \otimes_{\sigma'} \mathcal{M} \cong \sum_{s \geq 0}^{\oplus} \mathfrak{L}^s(\mathcal{M}).$$

In these notations we have $G_k = \mathfrak{L}^k(G_0) \cong E^{\otimes k} \otimes_\sigma G_0 \cong \sum_{l \geq k}^{\oplus} E^{\otimes l} \otimes_\pi H$, with H as the wandering subspace and $\sigma' = \pi$.

3 Generalized inner-outer factorization

In this section we describe a general version of the theory of inner-outer factorization for an arbitrary element $g \in \mathcal{F}(E) \otimes_\pi H$ and arbitrary elements of commutant $\rho(H^\infty(E))'$, and then we deduce some natural version of factorization of elements of $\rho(H^\infty(E))$, where

$\rho = \rho_\pi$ denotes the representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\pi H$, induced by the faithful normal representation π. Although most of our constructions are correct in the general case we assume in the following that the space H of the representation π is separable (see Remark 3.14).

Before we start let S be a unilateral shift acting on the Hilbert space H and let $\mathcal{M} \subset H$ be an S-invariant subspace. Write $\mathcal{M}_0 := \mathcal{M} \ominus S(\mathcal{M})$ and $H_0 := H \ominus S(H)$ for the wandering subspaces of $S|_{\mathcal{M}}$ and of S correspondingly. Then one of the main points in the proofs of the classical theorems of Beurling, Halmos and Lax on invariant subspaces of S is that $\dim \mathcal{M}_0 \leq \dim H_0$.

In our situation let us consider $G = \mathcal{F}(E) \otimes_\pi H$ as the left $H^\infty(E)$-module with the action defined by $X \cdot g := \rho_\pi(X)g$, for any $X \in H^\infty(E)$ and $g \in G$. Thus, in this language every $\rho_\pi(H^\infty(E))$-invariant subspace $\mathcal{M} \subseteq G$ defines an $H^\infty(E)$-submodule in G. Note that in this case $End(G)$ - the set of all the endomorphisms of this module is nothing but $\rho_\pi(H^\infty(E))'$. The covariant representation (V, σ), associated with the induced representation ρ_π, defines the generalized shift $\mathfrak{L}$. Hence, we need to compare the wandering subspaces $G \ominus \mathcal{L}(G)$ and $\mathcal{M}_0 := \mathcal{M} \ominus \mathcal{L}(\mathcal{M})$. More precisely, we need to compare the representations of M on H and on the $\mathcal{M}_0$. This is done in the following proposition

Proposition 3.1. *[8, Proposition 4.1] Let $\mathcal{M}$ be a $\rho_\pi(H^\infty(E))$-invariant subspace of $\mathcal{F}(E) \otimes_\pi H$ and let (V, σ) be the associated covariant representation of (E, M). If $\mathcal{M}_0 = \mathcal{M} \ominus \mathfrak{L}(M)$ is the $\mathfrak{L}$- wandering subspace in $\mathcal{F}(E) \otimes_\pi H$, then the restriction $\sigma|_{\mathcal{M}_0}$ is unitarily equivalent to a subrepresentation of π if and only if there is a partial isometry in $\rho_\pi(H^\infty(E))'$ with final space $\mathcal{M}$*

As the induced covariant representation (V, σ) is a natural generalization of a pure isometry, that is of a shift operator, a partial isometry in $\rho_\pi(H^\infty(E))'$ was called an inner operator, [8]. In our work we shall generalize this definition, and shall use this term for a suitable isometric operator which intertwines representations of M.

In fact we use the modules's language only to emphasize the analogy with the classical theory of shifts. Instead of this we shall constantly use the language of generalized shift $\mathfrak{L}$, associated with the induced covariant representation (V, σ).

3.1 Inner-outer factorization of elements of $\mathcal{F}(E) \otimes_\pi H$

We turn to the inner-outer factorization of a vector in $\mathcal{F}(E) \otimes_\pi H$. To this end we prove a Beurling type theorem for a cyclic (V, σ)-invariant subspace generated by this vector, i.e. for subspaces of the form $\mathcal{M}_g = \overline{\rho(H^\infty(E))g}$, where $g \in G_0 := \mathcal{F}(E) \otimes_\pi H$ is arbitrary.

Write P_g for the projection $P_{\mathcal{M}_g}$ onto $\mathcal{M}_g$. Clearly, $\mathcal{M}_g$ is a $\rho(H^\infty(E))$-invariant subspace, $P_g \in \sigma(M)$ and the restriction $(V, \sigma)|_{\mathcal{M}_g}$ is an induced isometric covariant representation, as follows from [8, Proposition 2.11]. In particular, $\mathcal{M}_g \in Lat(\sigma(M))$ and the subspace $\mathfrak{L}(\mathcal{M}_g)$ is well defined. Set

$$\mathcal{N}_g := \mathcal{M}_g \ominus \mathfrak{L}(\mathcal{M}_g). \tag{4}$$

By Q_g we denote the orthogonal projection of G_0 on $\mathcal{N}_g$. Then $Q_g = P_g - L(P_g)$ is the wandering projection associated with the restricted representation $(V,\sigma)|\mathcal{M}_g$. Since $L^k(Q_g) \perp L^s(Q_g)$, $k \neq s$, and since $(V,\sigma)|\mathcal{M}_g$ is induced we obtain the decomposition $\mathcal{M}_g = \sum_{k\geq 0}^{\oplus} L^k(Q_g)\mathcal{M}_g$. Equivalently, $\mathfrak{L}^k(\mathcal{N}_g) \perp \mathfrak{L}^s(\mathcal{N}_g)$, $k \neq s$, and

$$\mathcal{M}_g = \mathcal{N}_g \oplus \mathfrak{L}(\mathcal{N}_g) \oplus \cdots .$$

We set $g_0 := Q_g g \in \mathcal{N}_g$.

Lemma 3.2. $\mathcal{N}_g = \overline{\sigma(M)g_0}$.

Proof.

Since $\sigma(a)g_0 \in \mathcal{N}_g$ for each $a \in M$, then $\overline{\sigma(M)g_0} \subset \mathcal{N}_g$. Let $z \in \mathcal{N}_g \ominus \overline{\sigma(M)g_0}$. Then $z \perp \sigma(a)g_0$ for each $a \in M$ and in particular $z \perp g_0$. Write $g = g_0 + (g - g_0)$. Since $g - g_0 \in \mathfrak{L}(\mathcal{M}_g)$, we get $z \perp g - g_0$. For each $k \geq 1$, $V^{\otimes k}(\xi)g \in \mathfrak{L}^k(\mathcal{M}_g) \subset \mathcal{M}_g \ominus \mathcal{N}_g$. So, $z \perp V^{\otimes k}(\xi)g$ for each $k \geq 1$, $\xi \in E^{\otimes k}$. It follows that $z \perp \mathcal{M}_g$ and then $z = 0$, $\forall z \in \mathcal{N}_g \ominus \overline{\sigma(M)g_0}$, i.e. $\mathcal{N}_g = \overline{\sigma(M)g_0}$. $\square$

Remark 3.3. It is easy to see that every element of the form $\xi \otimes h \in E^{\otimes k} \otimes_\pi H$, for every $k \geq 1$ and every $\xi \in E^{\otimes k}$, is a wandering vector.

For each $a \in M$ we set

$$\tau(a) = \langle \sigma(a)g_0, g_0 \rangle = \langle (\phi_\infty(a) \otimes I)g_0, g_0 \rangle. \tag{5}$$

This defines a positive ultraweakly continuous linear functional τ on M. Since π is assumed to be faithful, we can view τ as defined on $\pi(M) \subset B(H)$.

Hence, there is a sequence $\{h_i\} \subset H$ with $\sum_i \|h_i\|^2 \leq \infty$, such that

$$\tau(a) = \sum_i \langle \pi(a)h_i, h_i \rangle = \langle \sigma(a)g_0, g_0 \rangle. \tag{6}$$

This sequence $\{h_i\}$ can be viewed as an element of the space $H^{(\infty)} = H \oplus H \oplus ...$ and we write $h_\tau = \{h_i\}$ for it to indicate that it is defined by the functional τ. For each $a \in M$ we define an operator (ampliation of π) $\hat{\pi}(a) = diag(\pi(a)) \in B(H^{(\infty)})$, acting on $H^{(\infty)}$ by: $\hat{\pi}(a)k = \{\pi(a)k_i\}$ where $k = \{k_i\} \in H^{(\infty)}$. Then, for h_τ we have

$$\tau(a) = \langle \hat{\pi}(a)h_\tau, h_\tau \rangle = \sum_i \langle \pi(a)h_i, h_i \rangle = \langle \sigma(a)g_0, g_0 \rangle.$$

Set

$$K_\tau := \overline{\hat{\pi}(M)h_\tau} \subseteq H^{(\infty)}, \tag{7}$$

and define the operator $w_0 : H^{(\infty)} \to \mathcal{N}_g$, by

$$\hat{\pi}(a)h_\tau \mapsto (\phi_\infty(a) \otimes I)g_0 = \sigma(a)g_0, \tag{8}$$

and $w_0 = 0$ on $H^{(\infty)} \ominus K_\tau$. Since $\langle \hat{\pi}(a)h_\tau, h_\tau \rangle = \langle \sigma(a)g_0, g_0 \rangle$, w_0 is a well defined partial isometry from $H^{(\infty)}$ onto $\mathcal{N}_g$. Taking $a = 1 \in M$, we get $w_0 h_\tau = g_0$, and we see that

$$w_0(\hat{\pi}(a)h_\tau) = \sigma(a)g_0 = \sigma(a)w_0(h_\tau).$$

Since the sets $\{\hat{\pi}(a)h_\tau : a \in M\}$ and $\{\sigma(a)g_0 : a \in M\}$ are dense in K_τ and $\mathcal{N}_g$ respectively we obtain that w_0 intertwines $\hat{\pi}$ and σ:

$$w_0\hat{\pi}(a) = \sigma(a)w_0, \ \forall a \in M. \tag{9}$$

We conclude:

Proposition 3.4. *The operator w_0 is a partial isometry intertwining $\hat{\pi}$ and σ, with initial subspace K_τ, and with $\mathcal{N}_g$ as final subspace.*

Now let us consider the space $G_0^{(\infty)} = G_0 \oplus G_0 \oplus ...$ and identify it with the space $\mathcal{F}(E) \otimes_{\hat{\pi}} H^{(\infty)}$. As usual we identify $M \otimes_{\hat{\pi}} H^{(\infty)}$ with $H^{(\infty)}$ and set $\hat{\mathfrak{L}}(H^{(\infty)}) := \mathfrak{L}(H) \oplus \mathfrak{L}(H) \oplus ... = \mathfrak{L}(H)^{(\infty)}$. Hence, for $G_0^{(\infty)}$ we can write the decomposition

$$G_0^{(\infty)} = H^{(\infty)} \oplus \hat{\mathfrak{L}}(H^{(\infty)}) \oplus \hat{\mathfrak{L}}^2(H^{(\infty)}) \oplus ... \tag{10}$$

Write $\hat{\rho}$ for the induced representation $\rho_{\hat{\pi}}$ of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_{\hat{\pi}} H^{(\infty)}$, $X \mapsto X \otimes I_{H^{(\infty)}}$, $X \in H^\infty(E)$. Thus, $\hat{\rho}$ is an ampliation of the induced representation ρ on $\mathcal{F}(E) \otimes_\pi H$. Then the associated isometric covariant representation of E on $\mathcal{F}(E) \otimes_{\hat{\pi}} H^{(\infty)}$ is the pair $(\hat{V}, \hat{\sigma})$ where $\hat{V}(\xi) = T_\xi \otimes I_{H^{(\infty)}}$ and $\hat{\sigma}(a) = \phi_\infty(a) \otimes I_{H^{(\infty)}}$. We call it an ampliation of (V, σ). Similarly we define the covariant representations $(\hat{V}^{\otimes k}, \hat{\sigma})$ and the associated operators $(\hat{V}_k)^\sim$, $k \geq 1$.

For each $k \geq 0$ we identify $\hat{\mathfrak{L}}^k(K_\tau)$ with $E^{\otimes k} \otimes_{\hat{\pi}} K_\tau$ and $\mathfrak{L}^k(\mathcal{N}_g)$ with $E^{\otimes k} \otimes_\sigma \mathcal{N}_g$. So, $\sum_{k\geq 0} \hat{\mathfrak{L}}^k(K_\tau) = \sum_{k\geq 0} E^{\otimes k} \otimes_{\hat{\pi}} K_\tau = \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau$ and $\mathcal{M}_g = \sum_{k\geq 0} E^{\otimes k} \otimes_\sigma \mathcal{N}_g = \mathcal{F}(E) \otimes_\sigma \mathcal{N}_g$

Write σ' for the restriction $\sigma|_{\mathcal{N}_g}$. Consider the restriction $\rho_\pi(H^\infty(E))|_{\mathcal{M}_g}$ and the induced representation $\sigma'^{\mathcal{F}(E)}(H^\infty(E))$ of $H^\infty(E)$ on $\mathcal{M}_g$. Then for every $k \geq 0$ and $\xi \otimes z \in E^{\otimes k} \otimes_\sigma \mathcal{N}_g$ we have

$$\rho_\pi(\phi_\infty(a))(\xi \otimes z) = (\phi_k(a) \otimes I_H)(\xi \otimes z) = (\phi(a)\xi) \otimes z = (\phi_k(a) \otimes I_{\mathcal{N}_g})(\xi \otimes z),$$

and

$$\rho_\pi(T_\theta)(\xi \otimes z) = (T_\theta \otimes I_H)(\xi \otimes z) = \theta \otimes \xi \otimes z = (T_\theta \otimes I_{\mathcal{N}_g})(\xi \otimes z).$$

Thus, the representation $\rho_\pi(H^\infty(E))|_{\mathcal{M}_g}$ is equal to the representation $\rho_{\sigma'}(H^\infty(E)) = \sigma'^{\mathcal{F}(E)}(H^\infty(E))$.

Using the fact that $\{\hat{\pi}(M)h_\tau\}$ is dense in K_τ and $\{\sigma(M)g_0\}$ is dense in $\mathcal{N}_g$ we define for every $k \geq 0$ the operator:

$$w_k : E^{\otimes k} \otimes_{\hat{\pi}} K_\tau \to E^{\otimes k} \otimes_\sigma \mathcal{N}_g,$$

by $\xi \otimes \hat{\pi}(a)h_\tau \mapsto \xi \otimes \sigma(a)g_0$, $\xi \in E^{\otimes k}$, $a \in M$. Since $\{\xi \otimes \hat{\pi}(a)h_\tau\}$ and $\{\xi \otimes \sigma(a)g_0\}$ span $E^{\otimes k} \otimes_{\hat{\pi}} K_\tau$ and $E^{\otimes k} \otimes_\sigma \mathcal{N}_g$ respectively, the operator w_k is well defined.

For $k = 0$ we have already showed that w_0 is an isometry from K_τ onto $\mathcal{N}_g$ that intertwines the representations $\hat{\pi}$ and σ.

Proposition 3.5. *The operator $w_k : E^{\otimes k} \otimes_{\hat{\pi}} K_\tau \to E^{\otimes k} \otimes_\sigma \mathcal{N}_g$ is a well defined isometry that intertwines the representation $\hat{\sigma}(\cdot)|_{E^{\otimes k} \otimes_{\hat{\pi}} K_\tau}$ and $\sigma(\cdot)|_{E^{\otimes k} \otimes_\pi H}$*

Proof. Let $\xi_i \otimes \hat{\pi}(a_i)h_\tau$, $i = 1, 2$, be in $E^{\otimes k} \otimes_{\hat{\pi}} K_\tau$, and $w_k(\xi_i \otimes \hat{\pi}(a_i)h_\tau) = \xi_i \otimes \sigma(a_i)g_0$. Denoting $c = \langle \xi_1, \xi_2 \rangle$ we obtain

$$\langle \xi_1 \otimes \hat{\pi}(a_1)h_\tau, \xi_2 \otimes \hat{\pi}(a_2)h_\tau \rangle = \langle \hat{\pi}(a_2)^* \hat{\pi}(c)^* \hat{\pi}(a_1)h_\tau, h_\tau \rangle = \langle \hat{\pi}(a_2^* c^* a_1)h_\tau, h_\tau \rangle.$$

Similarly,

$$\langle \xi_1 \otimes \sigma(a_1)g_0, \xi_2 \otimes \sigma(a_2)g_0 \rangle = \langle \sigma(a_2^* c^* a_1)g_0, g_0 \rangle,$$

so, w_k is an isometry.

Let $\xi \otimes k \in E^{\otimes k} \otimes_{\hat{\pi}} K_\tau$. Then $w_k(\xi \otimes k) = \xi \otimes z \in E^{\otimes k} \otimes_\sigma \mathcal{N}_g$, and

$$w_k((\phi_k(a) \otimes I_{K_\tau})(\xi \otimes k)) = w_k((\phi(a)\xi) \otimes k) = (\phi(a)\xi) \otimes z.$$

But

$$(\phi(a)\xi) \otimes z = (\phi_k(a) \otimes I_{\mathcal{N}_g})(\xi \otimes z) = (\phi_k(a) \otimes I_{\mathcal{N}_g})w_k(\xi \otimes k).$$

This proves the intertwining

$$w_k((\phi_k(a) \otimes I_{K_\tau})(\xi \otimes k)) = (\phi_k(a) \otimes I_{\mathcal{N}_g})w_k(\xi \otimes k).$$

$\square$

From the definition of the generalized powers we see that each w_k is associated with w_0 by the identity $V^{\otimes k}(\xi)w_0 = w_k \hat{V}^{\otimes k}(\xi)$, $\xi \in E^{\otimes k}$.

Now we set

$$W = \sum_k w_k : \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau \to \mathcal{F}(E) \otimes_\pi H. \tag{11}$$

It follows from the Propositions 3.4 and 3.5 that W is a well defined isometry and its image is $\mathcal{M}_g$.

Remark 3.6. It is obvious from the definition of w_k that $w_k(E^{\otimes k} \otimes_{\hat{\pi}} K_\tau) = E^{\otimes k} \otimes_{\hat{\sigma}} w_0(K_\tau)$ Fix $x \in \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau$ of the form $x = \xi \otimes k$, $\xi \in \mathcal{F}(E)$, and $k \in K_\tau$. Then we can write $Wx = W(\xi \otimes k) = \xi \otimes w_0 k$. Hence, $W = I_{\mathcal{F}(E)} \otimes w_0$.

Proposition 3.7. *The operator W is an isometry from $\tilde{K} := \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau \subset G^{(\infty)}$ into $\mathcal{F}(E) \otimes_\pi H$ with $\mathcal{M}_g$ as a final subspace. Further, W intertwines the representations $\hat{\rho}|_{\tilde{K}}$ and $\rho|_{\mathcal{M}_g}$ of the algebra $H^\infty(E)$:*

$$W\hat{\rho}(X)|_{\tilde{K}} = \rho(X)W. \tag{12}$$

for every $X \in H^\infty(E)$.

Proof. Its remains to show only the intertwining property. To show it, it is enough to show that (12) holds for the generators $\{T_\xi, \phi_\infty(a) : \xi \in E, a \in M\}$ of the Hardy algebra.

Since $W|_{E^{\otimes k}\otimes_{\hat{\pi}} K_\tau} = w_k$, the equality $W\hat{\rho}(\phi_\infty(a)) = \rho(\phi_\infty(a))W$, $a \in M$, follows form Proposition 3.5.

Now let $X = T_\xi$. Then $\hat{\rho}(T_\xi) = T_\xi \otimes I_{H^{(\infty)}}$ and $\rho(T_\xi) = T_\xi \otimes I_H$. Fix $\eta \otimes k \in \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau$, then using the previous remark we obtain
$W(T_\xi \otimes I_{H^{(\infty)}})(\eta \otimes k) = W(\xi \otimes \eta \otimes k) = \xi \otimes \eta \otimes w_0 k = (T_\xi \otimes I_{\mathcal{N}_g})(\eta \otimes w_0 k) = (T_\xi \otimes I_H)W(\eta \otimes k).$

□

We obtained an isometry $W : \tilde{K} = \mathcal{F}(E) \otimes_{\hat{\pi}} K_\tau \to \mathcal{F}(E) \otimes_\pi H$ with final subspace $\mathcal{M}_g = \mathcal{F}(E) \otimes_\sigma \mathcal{N}_g$ that intertwines the induced representations $\hat{\rho}$ and ρ of Hardy algebra $H^\infty(E)$. In the paper [8], partial isometries that lies in $\pi^{\mathcal{F}(E)}(\mathcal{T}_+(E))'$ are called inner operators. In our case the isometry W acts between different spaces, but intertwining $\hat{\rho}$ and ρ. So, it is natural to call such operators *inner* operators.

We present here the general definition

Definition 3.8. Given two normal representations π and μ of M on Hilbert spaces H and K respectively.

1) An isometry $W : \mathcal{F}(E) \otimes_\mu K \to \mathcal{F}(E) \otimes_\pi H$ will be called an inner operator if

(a) $K \subseteq H^{(\infty)}$ is a $\hat{\pi}(M)$- invariant subspace of $H^{(\infty)}$, where $\hat{\pi}$ be the ampliation of π on $H^{(\infty)}$ and $\mu = \hat{\pi}|_K$. In other words, K is an M-submodule of $H^{(\infty)}$ with respect to $\hat{\pi}$.

(b) $W\rho_\mu(X) = \rho_\pi(X)W, \quad X \in H^\infty(E)$.

2) A vector $y \in \mathcal{F}(E) \otimes_\mu K$ will be called outer if $\overline{\rho_\mu(H^\infty(E))y} = \mathcal{F}(E) \otimes_\mu K$.

This definition and Proposition 3.7 gives us the following Beurling type theorem for cyclic subspaces $\mathcal{M}_g$ that are considered as $\rho(H^\infty(E))$-modules.

Theorem 3.9. *Let $g \in G_0 = \mathcal{F}(E) \otimes_\pi H$ and let $\mathcal{M}_g = \overline{\rho((H^\infty(E))g}$ be a cyclic $\rho_\pi(H^\infty(E))$-submodule in G_0. Then there is a subspace $\mathcal{K} \subseteq H^{(\infty)}$ which is $\hat{\pi}(M)$-invariant and an inner operator $W : \mathcal{F}(E) \otimes_\mu \mathcal{K} \to G_0$ (where we write μ for $\hat{\pi}|_{\mathcal{K}}$) such that*

1)

$$\mathcal{M}_g = W(\mathcal{F}(E) \otimes_\mu \mathcal{K}). \tag{13}$$

2) The vector $y := W^* g$ is outer in $\mathcal{F}(E) \otimes_\mu \mathcal{K}$.

The outer vector $y = W^*g$ will be called the outer part of g. Thus, the outer part of an arbitrary $g \in G$ is an outer vector in the sense of Definition 3.8.

Definition 3.10. In the notation of the previous theorem, the equality

$$g = Wy, \tag{14}$$

will be called the inner-outer factorization of $g \in G_0$.

The first part of the following theorem was already proved:

Theorem 3.11. *Let $\pi : M \to B(H)$ be a faithful normal representation of W^*-algebra M on Hilbert space H. If $g \in \mathcal{F}(E) \otimes_\pi H$ then there is a M-submodule $\mathcal{K} \subset H^{(\infty)}$ with respect to the infinite ampliation $\hat{\pi}$ of π, an inner operator $W : \mathcal{F}(E) \otimes_\mu \mathcal{K} \to \mathcal{F}(E) \otimes_\pi H$, where $\mu = \hat{\pi}|_{\mathcal{K}}$, and an outer vector $y \in \mathcal{F}(E) \otimes_\mu \mathcal{K}$ such that $g = Wy$ is an inner-outer factorization of g.*

This factorization is unique in the following sense. Let $i = 1, 2$ and let $\mathcal{K}_i$ are two M-submodules in $H^{(\infty)}$ with respect to $\hat{\pi}$ and let $\mu_i = \hat{\pi}|_{\mathcal{K}_i}$ be two normal representations of M on $\mathcal{K}_i$. Suppose further that $W_i : \mathcal{F}(E) \otimes_{\mu_i} \mathcal{K}_i \to \mathcal{F}(E) \otimes_\pi H$ are inner operators and $y_i \in \mathcal{F}(E) \otimes_{\mu_i} \mathcal{K}_i$ are outer vectors such that $g = W_1 y_1 = W_2 y_2$. Then there is a unitary $U : \mathcal{F}(E) \otimes_{\mu_1} \mathcal{K}_1 \to \mathcal{F}(E) \otimes_{\mu_2} \mathcal{K}_2$ such that $Uy_1 = y_2$ and the equality $U\rho_{\mu_1}(X) = \rho_{\mu_2}(X)U$ holds for every $X \in H^\infty(E)$.

Proof. It remains to prove the uniqueness part. Let

$$W_i : \mathcal{F}(E) \otimes_{\mu_i} \mathcal{K}_i \to \mathcal{F}(E) \otimes_\pi H,$$

where μ_i, $\mathcal{K}_i$, y_i, $i = 1, 2$, are as in the statement of the theorem. Then $W_i y_i = g$ and

$$W_i \rho_{\mu_i}(X) = \rho_\pi(X) W_i, \quad X \in H^\infty(E), \ i = 1, 2. \tag{15}$$

Since $y_i = W_i^* g$ are outer in $\mathcal{F}(E) \otimes_{\mu_i} \mathcal{K}_i$, $i = 1, 2$, and since W_i have a common final subspace $\mathcal{M}_g \subset \mathcal{F}(E) \otimes_\pi H$, we get

$$W_1 \overline{\mu_1^{\mathcal{F}(E)}(H^\infty(E)) y_1} = \mathcal{M}_g = W_2 \overline{\mu_2^{\mathcal{F}(E)}(H^\infty(E)) y_2}.$$

Set $U := W_2^* W_1 : \mathcal{F}(E) \otimes_{\mu_1} \mathcal{K}_1 \to \mathcal{F}(E) \otimes_{\mu_2} \mathcal{K}_2$. Then U is a unitary operator and $Uy_1 = y_2$. Finally, from the intertwining relation (15) we obtain

$$W_1 \rho_{\mu_1}(X) W_1^* = W_2 \rho_{\mu_2}(X) W_2^*.$$

Hence,

$$U\rho_{\mu_1}(X) = \rho_{\mu_2}(X) U,$$

as we wanted. $\square$

Remark 3.12. Note that, in fact, the unitary U appearing in the proof can be thought of as a partial isometry in $\rho_{\hat{\pi}}(H^\infty(E))'$.

3.2 Inner-Outer factorization of elements of the algebra $\rho_\pi(H^\infty(E))'$

We shall now apply Theorem 3.11 to get an inner-outer factorization of an element of the commutant $\rho_\pi(H^\infty(E))'$.

First we consider the simple case when π is a cyclic representation of the algebra M, i.e. we assume that there is $h \in H$ such that $\overline{\pi(M)h} = H$.

Fix $S \in \rho_\pi(H^\infty(E))'$ and set $g := S(1 \otimes h) \in \mathcal{F}(E) \otimes_\pi H$, where $1 \otimes h \in M \otimes_\pi H$ and h is a π-cyclic vector in H.

Now form the subspace $\mathcal{M}_g = \overline{\rho_\pi(H^\infty(E))g} = \overline{\rho_\pi(H^\infty(E))S(1 \otimes h)}$. Since S is in the commutant of $\rho_\pi(H^\infty(E))$ and h is π-cyclic we obtain

$$\overline{\rho_\pi(H^\infty(E))S(1 \otimes h)} = \overline{S\rho_\pi(H^\infty(E))(1 \otimes h)} = \overline{S(\mathcal{F}(E) \otimes_\pi H)}.$$

Thus,

$$\mathcal{M}_g = \overline{S(\mathcal{F}(E) \otimes_\pi H)}.$$

By Theorem 3.11 there are a $\hat{\pi}$-invariant Hilbert subspace $\mathcal{K} \subseteq H^{(\infty)}$, an outer element $y \in \mathcal{F}(E) \otimes_\tau \mathcal{K}$, with $\tau = \hat{\pi}|_{\mathcal{K}}$, and an inner operator $W : \mathcal{F}(E) \otimes_\tau \mathcal{K} \to \mathcal{F}(E) \otimes_\pi H$ such that $Wy = g$ and $\mathcal{M}_g$ is the final subspace of W.

We set

$$Y := W^* S : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\tau \mathcal{K}. \tag{16}$$

Proposition 3.13. *1) $\overline{Y(\mathcal{F}(E) \otimes_\pi H)} = \mathcal{F}(E) \otimes_\tau \mathcal{K}$;*
2) $Y\rho_\pi(X) = \rho_\tau(X)Y$, $\forall X \in H^\infty(E)$.

Proof. 1) Since $S(\mathcal{F}(E) \otimes_\pi H)$ is dense in $\mathcal{M}_g$ and since W is an isometry with $\mathcal{M}_g$ as its final subspace, we obtain that $W^*S(\mathcal{F}(E) \otimes_\pi H)$ is dense in $\mathcal{F}(E) \otimes_\tau \mathcal{K}$.

2) Since $\rho_\pi(X)W = W\rho_\tau(X)$ and S is in commutant of $\rho_\pi(H^\infty(E))$, we have

$$Y\rho_\pi(X) = W^*S\rho_\pi(X) = W^*\rho_\pi(X)S = \rho_\tau(X)W^*S = \rho_\tau(X)Y,$$

$X \in H^\infty(E)$. $\square$

The operator Y will be called the outer part of S and the equality $S = WY$ we call the inner-outer factorization of the operator $S \in \rho_\pi(H^\infty(E))'$. The definition of the operator Y a priory depends on the choice of the cyclic vector h. Let $h' \in H$ be another cyclic vector, $\overline{\pi(M)h'} = H$, and set $g' := S(a \otimes h')$ and $\mathcal{M}_{g'} = \overline{\rho_\pi(H^\infty(E))S(1 \otimes h')}$. Then

$$\mathcal{M}_{g'} = \overline{S\rho_\pi(H^\infty(E))(1 \otimes h')} = \mathcal{M}_g.$$

Now, by Theorem 3.11, there are $\hat{\pi}$-invariant Hilbert subspace $\mathcal{K}' \subseteq H^{(\infty)}$, representation $\tau' = \hat{\pi}|_{\mathcal{K}'}$, the outer vector $y' \in \mathcal{F}(E) \otimes_{\tau'} \mathcal{K}'$ and an inner operator W' such that $W'y' = g'$. Then the corresponding outer part is $Y' = W'^*S$. The operators W and W' have a common final subspace $\mathcal{M}_g$ and we define $U := W^*W'$. Hence, the operator $U : \mathcal{F}(E) \otimes_{\tau'} \mathcal{K}' \to \mathcal{F}(E) \otimes_\tau \mathcal{K}$ is unitary such that $W' = WU$ and $U\rho_{\tau'}(X) = \rho_\tau(X)U$. The last intertwining

relation follows as in the proof of Theorem 3.11. Further, we have $Y = W^*S$, $Y' = W'^*S$ and then $S = WY = W'Y' = WUY'$. Thus, $Y = UY'$. This shows that the definition of Y does not depend on the choice of the cyclic element $h \in H$ up to the unitary operator U.

Any operator $Z : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\tau \mathcal{K}$ with dense range that intertwines the representations ρ_τ and ρ_π of $H^\infty(E)$, will be called an *outer* operator. Before we give the general definition we consider the general case of noncyclic representation π.

So let $\pi : M \to B(H)$ be, as usual, a faithful normal representation and let $S \in \rho_\pi(H^\infty(E))'$

Set $\mathcal{M} := \overline{S(\mathcal{F}(E) \otimes_\pi H)}$ and let $P_\mathcal{N} := P_\mathcal{M} - L(P_\mathcal{M})$ be a wandering projection with range $\mathcal{N}$. Then in terms of the shift $\mathfrak{L}$ we get the Wold decomposition $\mathcal{M} = \mathcal{N} \oplus \mathfrak{L}(\mathcal{N}) \oplus \mathfrak{L}^2(\mathcal{N}) \oplus \cdots$ that we can identify with

$$\mathcal{M} = \mathcal{N} \oplus (E \otimes_\sigma \mathcal{N}) \oplus (E^{\otimes 2} \otimes_\sigma \mathcal{N}) \oplus \cdots \tag{17}$$

Consider the restriction of $\sigma(M)|_\mathcal{N}$. Then $\mathcal{N}$ can be written as a direct sum $\sum_i^\oplus \mathcal{N}_i$ of $\sigma(M)|_\mathcal{N}$-cyclic subspaces $\mathcal{N}_i$ with cyclic vectors $g_i \in \mathcal{N}$, such that $\mathcal{N}_i = \overline{\sigma(M)g_i}$. Thus,

$$\mathcal{N} = \sum_i^\oplus \overline{\sigma(M)g_i}.$$

The representation (V, σ) is an isometric representation and the generalized powers $\tilde{V}_k : E^{\otimes k} \otimes_\sigma (\mathcal{F}(E) \otimes_\pi H) \to \mathcal{F}(E) \otimes_\pi H$ are isometric operators. It follows that if either $k \neq l$ or $i \neq j$ one has $E^{\otimes k} \otimes_\sigma (\sigma(M)g_i) \perp E^{\otimes l} \otimes_\sigma (\sigma(M)g_j)$.

Then the Wold decomposition (17) can be written as

$$\mathcal{M} = \sum_i^\oplus \overline{\sigma(M)g_i} \oplus (E \otimes_\sigma \sum_i^\oplus \overline{\sigma(M)g_i}) \oplus ... \oplus (E^{\otimes k} \otimes_\sigma \sum_i^\oplus \overline{\sigma(M)g_i}) \oplus ...$$

Rearranging terms we can write

$$\mathcal{M} = \mathcal{M}_{g_1} \oplus \mathcal{M}_{g_2} \oplus ... \oplus \mathcal{M}_{g_m} \oplus ...,$$

where $\mathcal{M}_{g_i} = \sum_k^\oplus E^{\otimes k} \otimes_\sigma \mathcal{N}_i = \overline{\rho_\pi(H^\infty(E))g_i}$ with $\mathcal{N}_i$ as a wandering subspace in $\mathcal{M}_{g_i}$ (and thus the cyclic vectors g_i are wandering). From now on we shall write $\mathcal{M}_i = \mathcal{M}_{g_i}$ and then $\mathcal{M} = \sum_i^\oplus \mathcal{M}_i$.

Since all $\mathcal{M}_i$ are pairwise orthogonal we may apply Theorem 3.11 for every i. So, for every i there is a $\hat{\pi}(M)$-invariant Hilbert subspace $\mathcal{K}_i \subseteq H^{(\infty)}$, a normal representation $\tau_i = \hat{\pi}|_{\mathcal{K}_i}$ of M on $\mathcal{K}_i$, an outer element $y_i \in \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i$ and an inner operator $W_i : \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i \to \mathcal{F}(E) \otimes_\pi H$ such that $g_i = W_i y_i$, the final subspace of W_i is $\mathcal{M}_i$ and $W_i \rho_{\tau_i}(X) = \rho_\pi(X) W_i$, $X \in H^\infty(E)$. Further, for every i we set $Y_i := W_i^* S$. The representation $\sigma(M)|_\mathcal{N}$ is cyclic when restricted to $\mathcal{N}_i$, hence by Theorem 3.13 the operator Y_i has a dense range in $\mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i$, intertwines the representations ρ_π and ρ_{τ_i} of $H^\infty(E)$, and it does not depend on the choice of the cyclic element up to some unitary operator U_i.

Remark 3.14. Each $\mathcal{K}_i$ is a $\hat{\pi}(M)$-invariant subspace of $H^{(\infty)}$. If we write n for the cardinality of the set of the cyclic vectors $\{g_i\}$, then, since H is separable, $n \leq \aleph_0$. Thus, identifying $H^{(\infty)}$ with $(H^{(\infty)})^{(n)}$ we can, and will, assume that $\{\mathcal{K}_i\}$ are pairwise orthogonal subspaces in $H^{(\infty)}$ and we write $\mathcal{K} = \sum_i^{\oplus} \mathcal{K}_i$. In this case the representation $\tau = \sum_i \tau_i$ is subrepresentation of $\hat{\pi}$ obtained by restricting $\hat{\pi}$ to the $\hat{\pi}(M)$-invariant subspace $\mathcal{K} \subseteq H^{(\infty)}$

In view of this remark, the operator $W := \sum_i W_i$ acts from the subspace $\mathcal{F}(E) \otimes_\tau \mathcal{K} \subseteq \mathcal{F}(E) \otimes_{\hat{\pi}} H^{(\infty)}$ into $\mathcal{F}(E) \otimes_\pi H$ and is an inner operator. We also write $Y := \sum_i Y_i : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\tau \mathcal{K}$, and it follows that $S = WY$.

Definition 3.15. In the above notations, each operator $Y : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\tau \mathcal{K}$ that has a dense range and such that $Y\rho_\pi(X) = \rho_\tau(X)Y$, for every $X \in H^\infty(E)$, will be called an outer operator.

If $S \in \rho_\pi(H^\infty(E))'$ then every factorization of S is of the form

$$S = WY, \tag{18}$$

where Y is an outer operator with a dense range in $\mathcal{F}(E) \otimes_\tau \mathcal{K}$, and W is an inner operator from $\mathcal{F}(E) \otimes_\tau \mathcal{K}$ into $\mathcal{F}(E) \otimes_\pi H$ will be called an inner-outer factorization of S. The operator Y in such factorization will be called the outer part of S. We write also Y_S for Y.

The outer part $Y_S = W^*S$ of $S \in \rho_\pi(H^\infty(E))'$ is indeed an outer operator since

$$\rho_\tau(X)Y_S = \rho_\tau(X)W^*S = W^*\rho_\pi(X)S = W^*S\rho_\pi(X) = Y_S\rho_\pi(X).$$

We proved the existence part of the following theorem

Theorem 3.16. *Let $S \in \rho_\pi(H^\infty(E))'$. Then there exist a $\hat{\pi}$-invariant subspace $\mathcal{K} \subseteq H^{(\infty)}$, a normal representation $\tau = \hat{\pi}|_{\mathcal{K}}$ of M on $\mathcal{K}$, an inner operator $W : \mathcal{F}(E) \otimes_\tau \mathcal{K} \to \mathcal{F}(E) \otimes_\pi H$ and an outer operator $Y : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\tau \mathcal{K}$ such that $S = WY$.*

This factorization is unique in the following sense. If there is other $\hat{\pi}$-invariant subspace $\mathcal{K}' \subseteq H^{(\infty)}$, a normal representation $\tau' = \hat{\pi}|_{\mathcal{K}'}$ of M on $\mathcal{K}'$, and if $S = W'Y'$, where $W' : \mathcal{F}(E) \otimes_{\tau'} \mathcal{K}' \to \mathcal{F}(E) \otimes_\pi H$ is an inner operator with final subspace $\mathcal{M} = \overline{S(\mathcal{F}(E) \otimes_\pi H)}$, and $Y' : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_{\tau'} \mathcal{K}'$, is an outer operator, then there exist a unitary operator $U : \mathcal{F}(E) \otimes_\tau \mathcal{K} \to \mathcal{F}(E) \otimes_{\tau'} \mathcal{K}'$ such that $W' = UW$ and $Y' = U^*Y$, and $U\rho_\tau(X) = \rho_{\tau'}(X)U$, $X \in H^\infty(E)$.

Proof. The existence is proved above. For the uniqueness set $U = W^*W'$. Since W and W' have a common final subspace, the operator U is unitary and $W' = UW$. From $W'Y' = S = WY$ we easily obtain that $Y' = U^*Y$. The intertwining property for U follows form the definition of U and from the intertwining properties of W and W'. As in inner-outer factorization of vector, the unitary U can be thought of as a partial isometry in $\rho_{\hat{\pi}}(H^\infty(E))'$. $\square$

Let $V \in \rho_\pi(H^\infty(E))'$ be a partial isometry and let $V = WY$ be its inner-outer factorization. In this case the outer part of Y is also a partial isometry with $\ker Y = \ker V$.

In the paper [8] Muhly and Solel proved Beurling Theorem for $\mathcal{T}_+(E)$-invariant subspaces. They considered the C^*-correspondence E and assumed that $\mathcal{T}_+(E)$ is represented by some isometric representation. In their proof they used an additional assumption of quasi-invariance of the representation π, [8, page 868]. J. Meyer in his Ph.D. Thesis [7] pointed out that if π is a faithful normal representation of a W^*-algebra M, E is a W^*-correspondence over M and ρ is the induced representation ρ_π of $H^\infty(E)$, then the quasi-invariance assumption is fulfilled. Hence, the theorem can be formulated as follows:

Theorem 3.17. *For every $\rho(H^\infty(E))$-invariant subspace $\mathcal{M}$ there exist a family of partial isometries $\{V_i\}_i \subset \rho(H^\infty(E))'$ such that ranges of V_i are pairwise orthogonal and $\mathcal{M} = \sum_i V_i(\mathcal{F}(E) \otimes_\pi H)$.*

Since H assumed to be separable, the family $(V_i)_i$ is at most countable. Now we apply Theorem 3.16 for each V_i to obtain an inner-outer decomposition $V_i = W_iY_i$, where $W_i : \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i \to \mathcal{F}(E) \otimes_\pi H$ is the inner operator corresponding to V_i. Set as above $\mathcal{K} = \sum_i^\oplus \mathcal{K}_i$ and $\tau = \sum_i^\oplus \tau_i$. Then $\mathcal{F}(E) \otimes_\tau \mathcal{K} = \sum_\iota \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i$ and write $W = \sum_i W_i$. Then the Beurling theorem of Muhly and Solel can be reformulated in the following way.

Theorem 3.18. *Let $\pi : M \to B(H)$ be a faithful normal representation and let $\rho_\pi : X \mapsto X \otimes I_H$ be the representation induced by π of the Hardy algebra $H^\infty(E)$. Further, let $\mathcal{M} \subseteq \mathcal{F}(E) \otimes_\pi H$ be a $\rho_\pi(H^\infty(E))$-invariant subspace. Then there exists a sequence of inner operators $W_i : \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i \to \mathcal{F}(E) \otimes_\pi H$ with pairwise orthogonal ranges $\{\mathcal{M}_i\}$ such that*

$$\mathcal{M} = W(\mathcal{F}(E) \otimes_\tau \mathcal{K}), \tag{19}$$

where $\mathcal{F}(E) \otimes_\tau \mathcal{K} = \sum_i^\oplus \mathcal{F}(E) \otimes_{\tau_i} \mathcal{K}_i$ and $W = \sum_i W_i$.

Remark 3.19. 1) The initial projections $V_i^*V_i$ also lie in the commutant $\rho_\pi(H^\infty(E))'$. Since $V_i = W_iY_i$, then these projections are $Y_i^*Y_i$.

2) Every $\rho_\pi(H^\infty(E))$-invariant subspace in $\mathcal{F}(E) \otimes_\pi H$ is a direct sum of a cyclic subspaces $\mathcal{M}_{g_i}$ for some $g_i \in \mathcal{F}(E) \otimes_\pi H$, $i \in \mathbb{N}$.

3.3 Factorization of elements of $\rho_\pi(H^\infty(E))$

In this subsection we use the concept of duality of W^*-correspondences to produce a natural factorization of an arbitrary element of $\rho_\pi(H^\infty(E))$. This concept was developed in [10, Section 3].

Let $\pi : M \to B(H)$ be a normal representation of M on a Hilbert space H. We put

$$E^\pi := \{\eta : H \to E \otimes_\pi H : \eta\pi(a) = (\phi(a) \otimes I_H)\eta, a \in M\}. \tag{20}$$

On the set E^π we define the structure of a W^*-correspondence over the von Neumann algebra $\pi(M)'$ putting $\langle \eta, \zeta \rangle := \eta^*\zeta$ for the $\pi(M)'$-valued inner product, $\eta, \zeta \in E^\pi$. It is

easy to check that $\langle \eta, \zeta \rangle \in \pi(M)'$. For the bimodule operations: $b \cdot \eta = (I \otimes b)\eta$, and $\eta \cdot c = \eta c$, where $b, c \in \pi(M)'$.

Definition 3.20. The W^*-correspondence E^π is called the π-dual of E.

Let $\iota : \pi(M)' \to B(H)$ be the identity representation. Then we can form $E^{\pi,\iota} := (E^\pi)^\iota$. So, $E^{\pi,\iota} = \{S : H \to E^\pi \otimes_\iota H : S\iota(b) = \iota^{E^\pi} \circ \phi_{E^\pi}(b)S, b \in \pi(M)'\}$. This is a W^*-correspondence over $\pi(M)'' = \pi(M)$.

In [10] it was proved that for every faithful normal representation π of a W^*-algebra M, every W^*-correspondence E over M is isomorphic to $E^{\pi,\iota}$. We give a short description of this isomorphism.

For $\xi \in E$ let $L_\xi : H \to E \otimes_\pi H$ be defined by $L_\xi = \xi \otimes h$, $h \in H$. Then L_ξ is a bounded linear map since $\|L_\xi h\|^2 \leq \|\xi\|^2 \|h\|^2$ and $L_\xi^*(\zeta \otimes h) = \pi(\langle \xi, \zeta \rangle)h$. For each $\xi \in E$ we define the map $\hat{\xi} : H \to E^\pi \otimes_\iota H$ by means of its adjoint:

$$\hat{\xi}^*(\eta \otimes h) = L_\xi^*(\eta(h)),$$

$\eta \otimes h \in E^\pi \otimes_\iota H$.

Theorem 3.21. *([10, Theorem 3.6]) If the representation π of M on H is faithful, then the map $\xi \mapsto \hat{\xi}$ just defined, is an isomorphism of the W^*-correspondences E and $E^{\pi,\iota}$.*

For every $k \geq 0$, let $U_k : E^{\otimes k} \otimes_\pi H \to (E^\pi)^{\otimes k} \otimes_\iota H$ be the map defined in terms of its adjoint by $U_k^*(\eta_1 \otimes ... \otimes \eta_n \otimes h) = (I_{E^{\otimes k-1}} \otimes \eta_1)...(I_E \otimes \eta_{k-1})\eta_k(h)$. It is proved in [10] that U_k is a Hilbert space isomorphism from $E^{\otimes k} \otimes_\pi H$ onto $(E^\pi)^{\otimes k} \otimes_\iota H$.

By Theorem 3.21, for every $k \geq 1$ the W^*-correspondence $E^{\otimes k}$ over M is isomorphic to the W^*-correspondence $(E^{\otimes k})^{\pi,\iota} \cong (E^{\pi,\iota})^{\otimes k}$. If $\xi \in E^{\otimes k}$ then the corresponding element $\widehat{\xi} \in (E^{\otimes k})^{\pi,\iota}$ is defined now by the formula

$$\widehat{\xi}^*(\eta_1 \otimes ... \otimes \eta_k \otimes h) = L_\xi^* U_k^*(\eta_1 \otimes ... \otimes \eta_k \otimes h),$$

where $L_\xi : h \mapsto \xi \otimes h$ is a bounded linear map from H to $E^{\otimes k} \otimes_\pi H$. Thus, we obtain

$$\hat{\xi} = U_k L_\xi, \quad \text{for}\, \xi \in E^{\otimes k}. \tag{21}$$

For the dual correspondence (π-dual to E) we can form the (dual) Fock space $\mathcal{F}(E^\pi)$, which is a W^*-correspondence over $\pi(M)'$, and the Hilbert space $\mathcal{F}(E^\pi) \otimes_\iota H$. Let us define $U := \sum_{k\geq 0}^{\oplus} U_k$. It follows that the map $U := \sum_{k\geq 0}^{\oplus} U_k$ is a Hilbert space isomorphism from $\mathcal{F}(E) \otimes_\pi H$ onto $\mathcal{F}(E^\pi) \otimes_\iota H$, and its adjoint acts on decomposable tensors by $U^*(\eta_1 \otimes ... \otimes \eta_n \otimes h) = (I_{E^{\otimes n-1}} \otimes \eta_1)...(I_E \otimes \eta_{n-1})\eta_n h$.

Definition 3.22. The map $U_\pi = U : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E^\pi) \otimes_\iota H$ will be called the Fourier transform determined by π.

Let $\pi : M \to B(H)$ be a faithful normal representation. Then there exists a natural isometric representation of $(E^\pi, \pi(M)')$ on $\mathcal{F}(E) \otimes_\pi H$ induced by π. Let $\nu : \pi(M)' \to B(\mathcal{F}(E) \otimes_\pi H)$ be a $*$-representation defined by $\nu(b) = I_{\mathcal{F}(E)} \otimes b$. Then ν is a faithful normal representation of the von Neumann algebra $\pi(M)'$ and by Theorem 2.3, $\pi^{\mathcal{F}(E)}(\mathcal{L}(\mathcal{F}(E)))' = \nu(\pi(M)') = \{I_{\mathcal{F}(E)} \otimes b : b \in \pi(M)'\}$. Given $\eta \in E^\pi$, for each $n \geq 0$ the operators $L_{\eta,n} : E^{\otimes n} \otimes_\pi H \to E^{\otimes n+1} \otimes_\pi H$ are defined by $L_{\eta,n}(\xi \otimes h) = \xi \otimes \eta h$, where we have identified $E^{\otimes n+1} \otimes_\pi H$ with $E^{\otimes n} \otimes_{\pi^E \circ \phi} (E \otimes_\pi H)$. Since $\|L_{\eta,n}\| \leq \|\eta\|$, we may define the operator $\Psi(\eta) : \mathcal{F}(E) \otimes_\pi H \to \mathcal{F}(E) \otimes_\pi H$ by $\Psi(\eta) = \sum_{k \geq 0}^{\oplus} L_{\eta,k}$. Thus we may think of $\Psi(\eta)$ as $I_{\mathcal{F}(E)} \otimes \eta$ on $\mathcal{F}(E) \otimes_\pi H$. It is easy to see that Ψ is a bimodule map. For the inner product, let $\eta_1, \eta_2 \in E^\pi$ and $\xi \otimes h, \zeta \otimes k \in E^{\otimes n} \otimes_\pi H$, then a simple calculation shows that

$$\langle \Psi(\eta_1)(\xi \otimes h), \Psi(\eta_2)(\zeta \otimes k) \rangle = \langle \xi \otimes h, \nu(\eta_1^* \eta_2)(\zeta \otimes k) \rangle,$$

so, (Ψ, ν) is an isometric representation of $(E^\pi, \pi(M)')$ on the Hilbert space $\mathcal{F}(E) \otimes_\pi H$. Combining the integrated form $\nu \times \Psi$ of (Ψ, ν) with the definition of the Fourier transform $U = U_\pi$ we obtain

$$U^* \iota^{\mathcal{F}(E^\pi)}(T_\eta) U = \Psi(\eta), \tag{22}$$

where $\eta \in E^\pi$ and T_η is the corresponding creation operator in $H^\infty(E^\pi)$, and

$$U^* \iota^{\mathcal{F}(E^\pi)}(\phi_{E^\pi,\infty}(b)) U = \nu(b), \tag{23}$$

where $b \in \pi(M)'$ and $\phi_{E^\pi,\infty}$ is the left action of $\pi(M)'$ on $\mathcal{F}(E^\pi)$. This equality can be rewritten as

$$U(I_{\mathcal{F}(E)} \otimes b) = (\phi_{E^\pi,\infty}(b) \otimes I_H) U. \tag{24}$$

Thus, the Fourier transform $U = U_\pi$ intertwines the actions of $\pi(M)'$ on $\mathcal{F}(E) \otimes_\pi H$ and on $\mathcal{F}(E^\pi) \otimes_\iota H$ respectively.

The following theorem identifies the commutant of the Hardy algebra represented by an induced representation.

Theorem 3.23. *([10], Theorem 3.9) Let E be a W^*-correspondence over M, and let $\pi : M \to B(H)$ be a faithful normal representation of M on a Hilbert space H. Write ρ_π for the representation $\pi^{\mathcal{F}(E)}$ of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\pi H$ induced by π, and write ρ^π for the representation of $H^\infty(E^\pi)$ on $\mathcal{F}(E) \otimes_\pi H$ defined by*

$$\rho^\pi(X) = U^* \iota^{\mathcal{F}(E^\pi)}(X) U, \tag{25}$$

with $X \in H^\infty(E^\pi)$. Then ρ^π is an ultraweakly continuous, completely isometric representation of $H^\infty(E^\pi)$ that extends the representation $\nu \times \Psi$ of $\mathcal{T}_+(E^\pi)$, and $\rho^\pi(H^\infty(E^\pi))$ is the commutant of $\rho_\pi(H^\infty(E))$, i.e. $\rho^\pi(H^\infty(E^\pi)) = \rho_\pi(H^\infty(E))'$.

Corollary 3.24. *([10], Corollary 3.10) In the preceding notation, $\rho_\pi(H^\infty(E))'' = \rho_\pi(H^\infty(E))$.*

Now we turn to the factorization of elements of $\rho_\pi(H^\infty(E))$. It will be obtained as a corollary of Theorem 3.16.

Let $X \otimes I_H \in \rho_\pi(H^\infty(E))$ and set

$$\mathcal{M} := \overline{(X \otimes I_H)(\mathcal{F}(E) \otimes_\pi H)}.$$

Then $\mathcal{M}$ is $\rho_\pi(H^\infty(E))'$-invariant. Now let U_π be a Fourier transform defined by π. Then the subspace

$$\tilde{\mathcal{M}} := U_\pi \mathcal{M} \subseteq \mathcal{F}(E^\pi \otimes_\iota H)$$

is $\rho_\iota(H^\infty(E^\pi))$ - invariant, where by ρ_ι we denote the induced representation $\iota^{\mathcal{F}(E^\pi)}$ of $H^\infty(E^\pi)$ on $\mathcal{F}(E^\pi) \otimes_\iota H$. Set $\tilde{X} = U_\pi(X \otimes I_H)U_\pi^*$. Then $\tilde{X}$ is in the commutant of $\rho_\iota(H^\infty(E^\pi))$ (see Theorem 3.23) and

$$\tilde{\mathcal{M}} = \overline{U_\pi(X \otimes I_H)U_\pi^*(\mathcal{F}(E^\pi) \otimes_\iota H)} = \overline{\tilde{X}(\mathcal{F}(E^\pi) \otimes_\iota H)}.$$

Write $\hat{\iota}$ for the ampliation of ι on the space $H^{(\infty)}$. By Theorem 3.16 there is a $\hat{\iota}(\pi(M)')$-invariant subspace $\mathcal{L}$ in $H^{(\infty)}$, an inner operator

$$\tilde{W} : \mathcal{F}(E^\pi) \otimes_{\hat{\tau}} \mathcal{L} \to \mathcal{F}(E^\pi) \otimes_\iota H,$$

where $\hat{\tau} = \hat{\iota}|_{\mathcal{L}}$, with a final subspace $\tilde{\mathcal{M}}$, and an outer operator $\tilde{Y} = \tilde{W}^* \tilde{X}$, $\overline{\tilde{Y}(\mathcal{F}(E^\pi) \otimes_\iota H)} = \mathcal{F}(E^\pi) \otimes_{\hat{\tau}} \mathcal{L}$, such that $\tilde{X} = \tilde{W}\tilde{Y}$ is the inner-outer factorization of $\tilde{X}$.

Hence, $\tilde{X} = U_\pi(X \otimes I_H)U_\pi^* = \tilde{W}\tilde{Y}$, and

$$X \otimes I_H = U_\pi^* \tilde{W}\tilde{Y} U_\pi. \tag{26}$$

Theorem 3.25. *For every $X \in H^\infty(E)$ the operator $\rho_\pi(X) = X \otimes I_H$ can be factorized as*

$$X \otimes I_H = \mathcal{W}\mathcal{Y}, \tag{27}$$

where $\mathcal{W}$ and $\mathcal{Y}$ satisfy

1) $\mathcal{W}$ is a partial isometry from $\mathcal{F}(E^\pi) \otimes_{\hat{\iota}} H^{(\infty)}$ into $\mathcal{F}(E) \otimes_\pi H$ with intertwining relation

$$\mathcal{W}\rho_{\hat{\tau}}(S) = \rho^\pi(S)\mathcal{W}, \ \ S \in H^\infty(E^\pi).$$

2) $\mathcal{Y}$ acts from $\mathcal{F}(E) \otimes_\pi H$ into $\mathcal{F}(E^\pi) \otimes_{\hat{\iota}} H^{(\infty)}$ and satisfies the intertwining relation

$$\mathcal{Y}\rho^\pi(S) = \rho_{\hat{\tau}}(S)\mathcal{Y}, \ \ S \in H^\infty(E^\pi).$$

3) the initial subspace of $\mathcal{W}$ is the closure of the range of $\mathcal{Y}$.
This factorization is unique up to a multiplication by unitary.

Proof. In (26) set

$$\mathcal{W} = U_\pi^* \tilde{W} \text{ and } \mathcal{Y} = \tilde{Y} U_\pi.$$

We have seen that $\mathcal{W}$ is a partial isometry from $\mathcal{F}(E) \otimes_{\hat{\tau}} \mathcal{L}$ into $\mathcal{F}(E) \otimes_\pi H$ with the final subspace $\mathcal{M}$, and that $\mathcal{Y}$ is the operator from $\mathcal{F}(E) \otimes_\pi H$ and has a closed range in $\mathcal{F}(E) \otimes_{\hat{\tau}} \mathcal{L}$.

Since $\tilde{W}$ is inner, then $\tilde{W} \rho_{\hat{\tau}}(S) = \rho_\iota(S) \tilde{W}$ for every $S \in H^\infty(E^\pi)$. Now, $U_\pi^* \rho_\iota(S) = \rho^\pi(S) U_\pi^*$, where $\rho^\pi(S) = U_\pi(\iota^{\mathcal{F}(E^\pi)}(S) U_\pi^*$ is the representation of $H^\infty(E^\pi)$ on $\mathcal{F}(E) \otimes_\pi H$ defined in (25). Thus,

$$\mathcal{W} \rho_{\hat{\tau}}(S) = \rho^\pi(S) \mathcal{W}.$$

Similarly we can show that

$$\mathcal{Y} \rho^\pi(S) = \rho_\iota(S) \mathcal{Y}, \;\; \forall S \in H^\infty(E^\pi).$$

The uniqueness up to multiplication by unitary follows from the uniqueness of the inner-outer factorization $\tilde{X} = \tilde{W} \tilde{Y}$. □

Bibliography

[1] A. Arias, G. Popescu, Factorization and reflexivity on Fock space, Integral equations Operator Theory, 23 (1995), 268-286.

[2] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta. Math. 81 (1949), 239-255.

[3] K.R. Davidson, D.R. Pitts, Invariant subspaces and hyper-reflexivity for free semigroup algebras, Proc. London Math. Soc.,(3) 78,(1999), 401-430.

[4] D. W. Kribs, S. C. Power, "Free semigroupoid algebras", J. Ramanujan Math. Soc. 19 no. 2,(2004), 117-159.

[5] E.C. Lance, Hilbert C^* - modules : A toolkit for operator algebraist, London, Math.Soc., Lecture Note Series 210, Cambridge Univ. Press, (1995).

[6] V. Manuilov, E. Troitsky, Hilbert C^*-modules, Translations of Mathematical Monographs, Vol.226, Amer. Math. Soc., Providence, 2005.

[7] J. R. Meyer, Noncommutative Hardy Algebras, Multipliers, and Quotients, Ph.D. Thesis, University of Iowa (2010).

[8] P. Muhly, B.Solel, Tensor algebras, Induced Representations and the Wold Decomposition, Canad. J. Math. 51(4), (1999) 850-880.

[9] P. Muhly, B.Solel, Tensor algebras over C^*-correspondence: Representations, Dilations, and C^*-Envelopes, J.Func.An, 158, (1998) 398-477.

[10] P. Muhly, B.Solel, Hardy algebras, W^*-correspondences and interpolation theory, Math.Ann.330, (2004) 353-415.

[11] P. Muhly, B.Solel, Representations of the Hardy algebras: Absolutely continuity, Intertwiners and Superharmonic operators, Integr. Equ. Oper. Theory 70 (2011), 151-203.

[12] N.Nikolskii, Operators, Functions, and Systems: an Easy reading. Vol. I&II, Parts A, B, C. Matheamical Surveys and Monographs, 92, AMS (2002).

[13] B. Sz.-Nagy, C. Foiaş, Harmonic Analysis of Operators on Hilbert Spaces, North-Holland Akad. Kiado (1970).

[14] W.L. Paschke, Inner product Modules over B^*-algebras, Transactions of the Amer. Math. Soc., Vol. 182, (1973), 443-468.

[15] M. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, in "Free Probability Theory", (D. Voiculescu, Ed.), Fields Instit.Comm., Vol. 12, Amer.Math.Soc., Providence, RI, (1997), pp.189-212.

[16] G. Popescu, Von Nemann inequality for $(B(H)^n)_1$, Math. Scand. 68 (1991), 292-304.

[17] G. Popescu, Multy-analytic operators and some factorization theorems, Indiana Univ. Math. J. 38 (1989), 693-710.

[18] G. Popescu, Multy-analytic operators, J. Operator Theory, 22 (1989), 51-71.

[19] G. Popescu, Multy-analytic operators on Fock spaces, Math. Ann. 303 (1995), 31-46.

[20] M. Rieffel, Induced representations of C^*-algebras, Adv. Math. 13 (1974), 176-257.

[21] S. Sakai, A characterization of W^*-algebras, Pacific J. Math.,6(1956) 763-773.

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